Impurity effect on the commensurate-incommensurate transition in the one-dimensional quantum sine-Gordon system ($g^2=4\pi$)



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# Impurity effect on the commensurate–incommensurate transition in the one-dimensional quantum sine–Gordon system ($g^2 = 4\pi$)

Hikaru Yamamoto

Institute of Physics, Faculty of Education, Shiga University, Otsu 520, Japan

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**Abstract.** We have studied the behaviour of the mean misfit and the specific heat of the one-dimensional quantum sine–Gordon system ($g^2 = 4\pi$) with a misfit parameter for two models of impurities. We have used a Fermi–Bose relation and a self-consistent $t$-matrix approximation for impurities. We have made numerical and analytical calculations for the mean misfit and the specific heat for various cases.

## 1. Introduction

In a previous paper [1] we have discussed a commensurate–incommensurate crossover in the one-dimensional quantum sine–Gordon system with a misfit parameter. The purpose of this paper is to study the effect of impurities. In order to take account of non-linear excitations, sine–Gordon models are widely used in many fields of condensed matter systems, for example, charge-density-wave (CDW) systems such as tetrathiafulvalene-tetracyanoquinodimethane (TTF-TCNQ) [2] and transition-metal dichalcogenides [3]. Also, impurities play an important role in many fields of condensed matter systems, for example, a dielectric system such as thiourea, in which the impurities stabilize the commensurate phases [4]. It is therefore a relevant and challenging issue to combine disorder (impurity) with non-linearity [5]. For example, commensurate–incommensurate transitions in two-dimensional adsorbed systems with quenched random impurities have been investigated using the replica and the Bethe *ansatz* methods [6] or the decimation method [7]. Moreover, two-dimensional $XY$ magnets or solid films with quenched random disorder have been studied using the renormalization-group approach [8, 9] or the replica and the renormalization-group methods [10]. The motion of interfaces and dislocations in disordered media has also been discussed [11].

Our model in this paper treats one-dimensional disorder rather than two-dimensional disorder. When the randomness of impurity positions is restricted only in the misfit direction, the two-dimensional classical problem [6] is mapped into our one-dimensional quantum problem.

We consider the following one-dimensional (1D) quantum sine–Gordon system with a misfit parameter $\hat{\mu}$ for the pure system:

$$H_{QS} = \tfrac{1}{2}\int \left[ p^2 + \left( \frac{\partial \phi}{\partial x} + \hat{\mu} \right)^2 - \frac{2\alpha_0}{g^2} \cos(g\phi) \right] \mathrm{d}x \tag{1}$$

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where $\phi(x)$ and $p(x)$ are, respectively, a Bose field and its canonical conjugate momentum. As in [12], we investigate two types of impurities. One is impurities that couple to the potential term:

$$H^1_{\rm imp} = -\int \left( U_0 \sum_j \delta(x - x_j) \cos[g\phi(x)] \right) {\rm d}x. \tag{2}$$

The other is impurities that couple to the gradient of the phase variable $\phi(x)$:

$$H^2_{\rm imp} = -\int \left( V_0 \sum_j \delta(x - x_j) \nabla\phi(x) \right) {\rm d}x. \tag{3}$$

For general values of $g^2$, equation (1) can be transformed into the massive Thirring model [13, 14], which includes the term of the fermion interaction. This massive Thirring model is exactly solved by the Bethe *ansatz* [15, 16]. At the special value of $g^2 = 4\pi$, the fermion interaction vanishes and the model reduces to a massive free-fermion model Hamiltonian:

$$H^0_{\rm MF} = \int \left[ -{\rm i} \left( \psi_1^\dagger \frac{\partial}{\partial x} \psi_1 - \psi_2^\dagger \frac{\partial}{\partial x} \psi_2 \right) - \tilde{\mu}(\psi_1^\dagger \psi_1 + \psi_2^\dagger \psi_2) + m_0(\psi_1^\dagger \psi_2 + \psi_2^\dagger \psi_1) \right] {\rm d}x \tag{4}$$

where $\psi_1$ and $\psi_2$ are fermion fields, $\tilde{\mu} = \sqrt{\pi}\hat{\mu}$ and $m_0 = a_1\alpha_0 c/4\pi$ ($c$ is the lattice constant and $a_1 \sim {\rm O}(1)$). In this case we may identify the soliton as a Fermi particle. The original non-linearity remains as a Fermi statistic.

Using the momentum-space operators $a_{j,k}$ defined as

$$\psi_j(x) = L^{-1/2} \sum_k a_{j,k} \exp({\rm i}kx) \tag{5}$$

we get

$$H^0_{\rm MF} = \sum_k A_k^\dagger (k\sigma_3 - \tilde{\mu}1 + m_0\sigma_1) A_k \tag{6}$$

where $A_k^\dagger = (a_{1,k}^\dagger, a_{2,k}^\dagger)$, $A_k$ is its adjoint column matrix and $\{\sigma_i\}$ are the Pauli matrices. This pure system has energy bands $E = \pm(k^2 + m_0^2)^{1/2}$. The impurity potential terms are written as

$$H^1_{\rm imp} = \sum_{k,k'} A_k^\dagger \left( \sum_j V_1 \exp[-{\rm i}(k - k')x_j]\sigma_1/L \right) A_{k'} \tag{7}$$

$$H^2_{\rm imp} = \sum_{k,k'} A_k^\dagger \left( \sum_j V_2 \exp[-{\rm i}(k - k')x_j]/L \right) A_{k'} \tag{8}$$

where $V_1 = a_1 U_0 c$ and $V_2 = \sqrt{\pi} V_0$.

We use a $2 \times 2$ matrix thermal Green function, $\hat{G}_{k,k'}(\tau) = -\langle T_\tau A_k(\tau); A_{k'}^\dagger(0)\rangle$, where $T_\tau$ is Wick's notation of the ordering operator for the imaginary time $\tau$. For the pure system ($H^0_{\rm MF}$) the Fourier transform of $\hat{G}^0_k(\tau)$ is given by

$$\hat{G}^0_k({\rm i}\omega_n) = [{\rm i}\omega_n + \tilde{\mu} - (k\sigma_3 + m_0\sigma_1)]^{-1} \tag{9}$$

where $\omega_n = 2\pi T(n + \frac{1}{2})$, with $n$ being integer, are the Matsubara frequencies.

In [12], we have considered the same sine–Gordon system, but with no misfit parameter, i.e. $\hat{\mu} = 0$. There we have used the self-consistent $t$-matrix approximation for impurities and derived the expression for the density of states. In our problem we make the same

approximation for impurities. Then the results are only to rewrite $i\omega_n \to i\omega_n + \tilde{\mu}$ in equations (15)–(18) of [12]. For the impurity potential $H^1_{imp}$, we obtain

$$i\tilde{\omega}_n = i\omega_n + \tilde{\mu} + \frac{n_i V_1^2 i\tilde{\omega}_n/2(\tilde{\omega}_n^2 + \tilde{m}_n^2)^{1/2}}{1 + V_1^2/4 + V_1\tilde{m}_n/(\tilde{\omega}_n^2 + \tilde{m}_n^2)^{1/2}} \tag{10a}$$

$$\tilde{m}_n = m_0 + n_i V_1 \frac{1 + V_1\tilde{m}_n/2(\tilde{\omega}_n^2 + \tilde{m}_n^2)^{1/2}}{1 + V_1^2/4 + V_1\tilde{m}_n/(\tilde{\omega}_n^2 + \tilde{m}_n^2)^{1/2}}. \tag{10b}$$

For the impurity potential $H^2_{imp}$, we obtain

$$i\tilde{\omega}_n = i\omega_n + \tilde{\mu} - n_i V_2 \frac{1 + V_2 i\tilde{\omega}_n/2(\tilde{\omega}_n^2 + \tilde{m}_n^2)^{1/2}}{1 - V_2^2/4 + V_2 i\tilde{\omega}_n/(\tilde{\omega}_n^2 + \tilde{m}_n^2)^{1/2}} \tag{11a}$$

$$\tilde{m}_n = m_0 + n_i V_2 \frac{-V_2\tilde{m}_n/2(\tilde{\omega}_n^2 + \tilde{m}_n^2)^{1/2}}{1 - V_2^2/4 + V_2 i\tilde{\omega}_n/(\tilde{\omega}_n^2 + \tilde{m}_n^2)^{1/2}}. \tag{11b}$$

Here $\tilde{\omega}_n$ and $\tilde{m}_n$ are, respectively, the renormalized frequency and the renormalized mass of the Green function averaged over the random spatial distribution of impurities:

$$\tilde{G}_k(i\omega_n) = [i\tilde{\omega}_n - (k\sigma_3 + \tilde{m}_n\sigma_1)]^{-1}. \tag{12}$$

The concentration of impurities is denoted by $n_i$.

Defining $U_n = \tilde{\omega}_n/\tilde{m}_n$, we obtain from (10*a*) and (10*b*) that

$$\omega_n - i\tilde{\mu} = U_n m_0 + \frac{n_i U_n}{A_1 + 1/(1 + U_n^2)^{1/2}} \tag{13}$$

and from (11*a*) and (11*b*) that

$$\omega_n - i\tilde{\mu} = U_n m_0 + \frac{-in_i V_2}{A_2 + iU_n/(1 + U_n^2)^{1/2}}. \tag{14}$$

Here $A_1 = 1/V_1 + V_1/4$ and $A_2 = 1/V_2 - V_2/4$. By replacing $iU_n \to U$ and $i\omega_n \to \omega$ in (13) and (14), we get analytically continued equations:

$$\omega + \tilde{\mu} = Um_0 + n_i U/[A_1 + 1/(1 - U^2)^{1/2}] \tag{15}$$

$$\omega + \tilde{\mu} = Um_0 + n_i/[A_2 + U/(1 - U^2)^{1/2}]. \tag{16}$$

In (15) and (16), $U$ is determined as a function of $\omega + \tilde{\mu}$, $U = U(\omega + \tilde{\mu})$.

The mean misfit (or winding number) $w$ is related to the fermion number $N = N(\tilde{\mu}, n_i, T)$,

$$w = (1/\sqrt{\pi})(\partial\phi/\partial x) = -(N - N_0)/L \tag{17}$$

where $N_0$ is the fermion number when $\tilde{\mu} = 0$ and $n_i = 0$. The fermion number $N$ can be expressed with use of the density of states $D(\omega$ and the Fermi distribution function $f(\omega)$:

$$N(\tilde{\mu}, n_i, T) = \frac{L}{\pi}\int D(\omega) f(\omega - \tilde{\mu})\,d\omega \tag{18}$$

$$D(\omega) = \mathrm{Im}(U(\omega)/[1 - U^2(\omega)]^{1/2}) \tag{19}$$

$$f(\omega) = 1/(e^{\omega/T} + 1). \tag{20}$$

At finite temperature $T$ the mean misfit is divided into three parts:

$$w_1 = -[N(\tilde{\mu}, n_i, T) - N(0, n_i, T)]/L \tag{21}$$

$$w_2 = -[N(0, n_i, T) - N(0, n_i, 0)]/L \tag{22}$$

$$w_3 = -[N(0, n_i, 0) - N(0, 0, 0)]/L. \tag{23}$$

The specific heat per unit length is written as

$$C = \frac{1}{\pi}\int D(\omega)\frac{(\omega-\tilde{\mu})^2}{2T^2}\frac{1}{1+\cosh[(\omega-\tilde{\mu})/T]}\,d\omega. \tag{24}$$

In the following sections we will discuss the above mean misfit and the specific heat of the system.

## 2. Impurity potential $H^1_{imp}$

### 2.1. Ground-state properties of the mean misfit

We will start from the discussion of the ground-state properties of the mean misfit. At $T = 0$, the fermion number is

$$N(\tilde{\mu}, n_i, T=0) = \frac{L}{\pi}\int_{-\Lambda}^{\tilde{\mu}} D(\omega)\,d\omega. \tag{25}$$

This is analytically calculated to become

$$N(\tilde{\mu}, n_i, T=0) = \frac{L}{\pi}\left\{\mathrm{Im}\left[\tilde{\mu}\frac{U}{(1-U^2)^{1/2}} - \frac{m_0}{(1-U^2)^{1/2}} - \frac{n_i}{2}\log\left(A_1 + \frac{1}{(1-U^2)^{1/2}}\right)^2\right] + \Lambda\right\}. \tag{26}$$

Here $U$ is determined by putting $\omega = 0$ in (15):

$$\tilde{\mu} = m_0 U + n_i U/[A_1 + 1/(1-U^2)^{1/2}]. \tag{27}$$

Then the mean misfit at $T = 0$ is written as

$$\frac{w}{m_0} = -\frac{1}{\pi}\mathrm{Im}\left[\frac{\tilde{\mu}}{m_0}\frac{U}{(1-U^2)^{1/2}} - \frac{1}{(1-U^2)^{1/2}} - \frac{n_i}{2m_0}\log\left(A_1 + \frac{1}{(1-U^2)^{1/2}}\right)^2\right]. \tag{28}$$

This behaviour is shown in figure 1($a$) for $V_1 > 0$ and in figures 2($a$) and 3($a$) for $V_1 < 0$.

*2.1.1. $V_1 > 0$.* For the repulsive impurity potential $V_1 > 0$, there is a gap energy $\omega_g$ for all concentration of impurities. For $|\omega| < \omega_g$, the density of states is zero. Near the gap energy, the density of states is expressed as

$$D(\omega) \sim \frac{1}{a^{1/2}(1-U_g^2)^{3/2}}\left(\frac{|\omega|-\omega_g}{m_0}\right)^{1/2} \tag{29}$$

where $U_g$ is the corresponding value of $\omega_g$, and $a = -[d^2(\omega/m_0)/dU^2]_{U=U_g}/2$. Integrating $D(\omega)$ with respect to $\omega$ from $\omega_g$ to $\tilde{\mu}$, we obtain the mean misfit:

$$\frac{w}{m_0} \sim -\frac{2}{3\pi}\frac{[(\tilde{\mu}-\omega_g)/m_0]^{3/2}}{a^{1/2}(1-U_g^2)^{3/2}}. \tag{30}$$

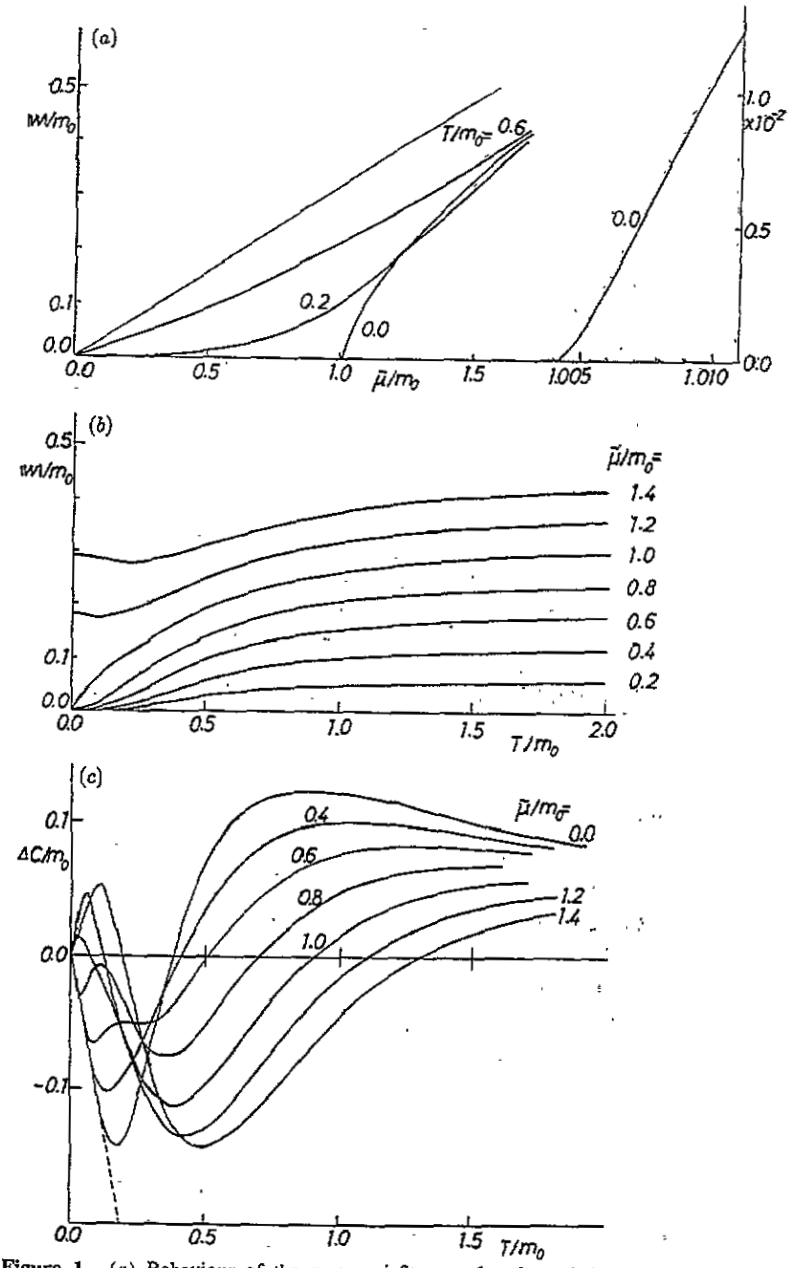


**Figure 1.** (*a*) Behaviour of the mean misfit as a function of the parameter $\tilde{\mu}$ at various temperatures $T/m_0 = 0$, 0.2 and 0.6 ($V_1 = 1.0$, $n_i/m_0 = 0.1$). The linear line $w = -\tilde{\mu}/\pi$ is that for the high-temperature limit. An enlarged scale is used in the right part to elucidate the critical behaviour of the mean misfit at $\tilde{\mu} \sim \omega_g$. (*b*) Behaviour of the mean misfit as a function of temperature for a set of parameters $\tilde{\mu}/m_0 = 0.2$, 0.4, 0.6, 0.8, 1.0, 1.2 and 1.4 ($V_1 = 1.0$, $n_i/m_0 = 0.1$). (*c*) Behaviour of the specific heat as a function of temperature for a set of parameters $\tilde{\mu}/m_0 = 0$, 0.4, 0.6, 0.8, 1.0, 1.2 and 1.4 ($V_1 = 1.0$, $n_i/m_0 = 0.1$). Here the specific heat for a massless boson gas with no impurities is subtracted, $\Delta C = C - \pi T/3$.

For low concentration of impurities, $U_g^2 \sim 1 - (n_i/m_0)^2$ and $a \sim (n_i/m_0)^{-2}/2$. Then the mean misfit is written as .

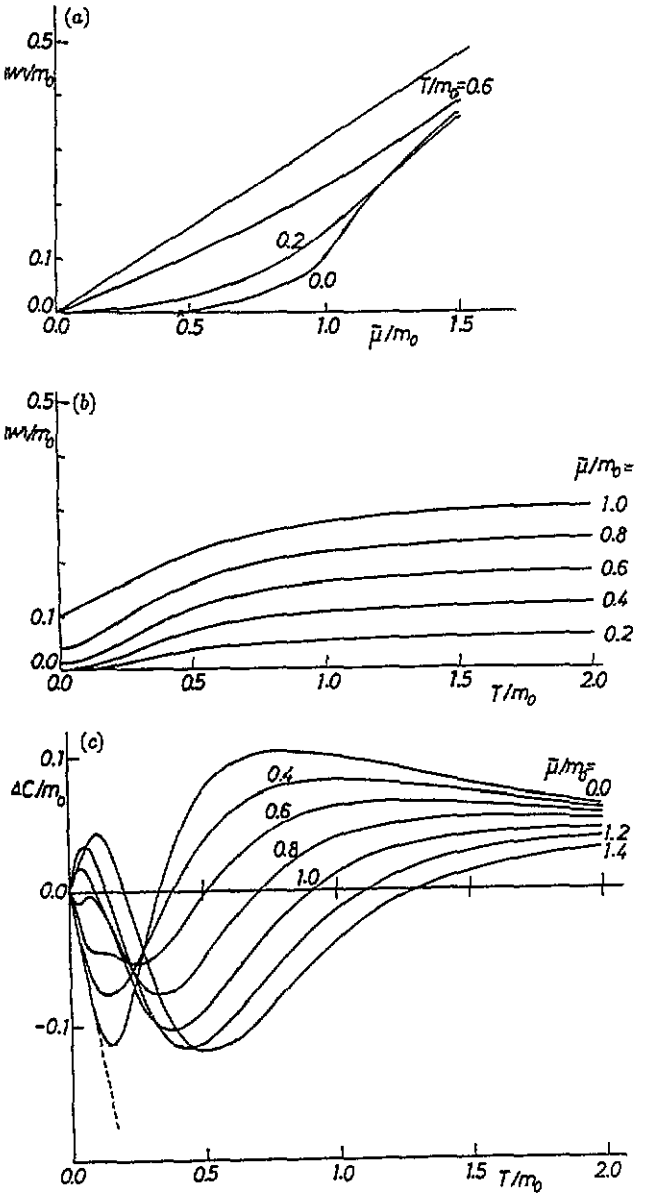


**Figure 2.** (*a*) Behaviour of the mean misfit as a function of the parameter $\tilde{\mu}$ at various temperatures $T/m_0 = 0$, 0.2 and 0.6 ($V_1 = -2/3$, $n_i/m_0 = 0.1$). The linear line $w = -\tilde{\mu}/\pi$ is that for the high-temperature limit. (*b*) Behaviour of the mean misfit as a function of temperature for a set of parameters $\tilde{\mu}/m_0 = 0.2$, 0.4, 0.6, 0.8 and 1.0 ($V_1 = -2/3$, $n_i/m_0 = 0.1$). (*c*) Behaviour of the specific heat as a function of temperature for a set of parameters $\tilde{\mu}/m_0 = 0$, 0.4, 0.6, 0.8, 1.0, 1.2 and 1.4 ($V_1 = -2/3$, $n_i/m_0 = 0.1$). Here the specific heat for a massless boson gas with no impurities is subtracted, $\Delta C = C - \pi T/3$.

$$\frac{w}{m_0} \sim -\frac{2\sqrt{2}}{3\pi}\left(\frac{n_i}{m_0}\right)^{-2}\left(\frac{\tilde{\mu}-\omega_g}{m_0}\right)^{3/2}. \tag{31}$$

The exponent of the deviation of $\tilde{\mu}$ from the gap energy is 3/2, which is different from that of the pure case, 1/2 [1, 17]. From the left part ($T = 0$) of figure 1(*a*) the exponent seems

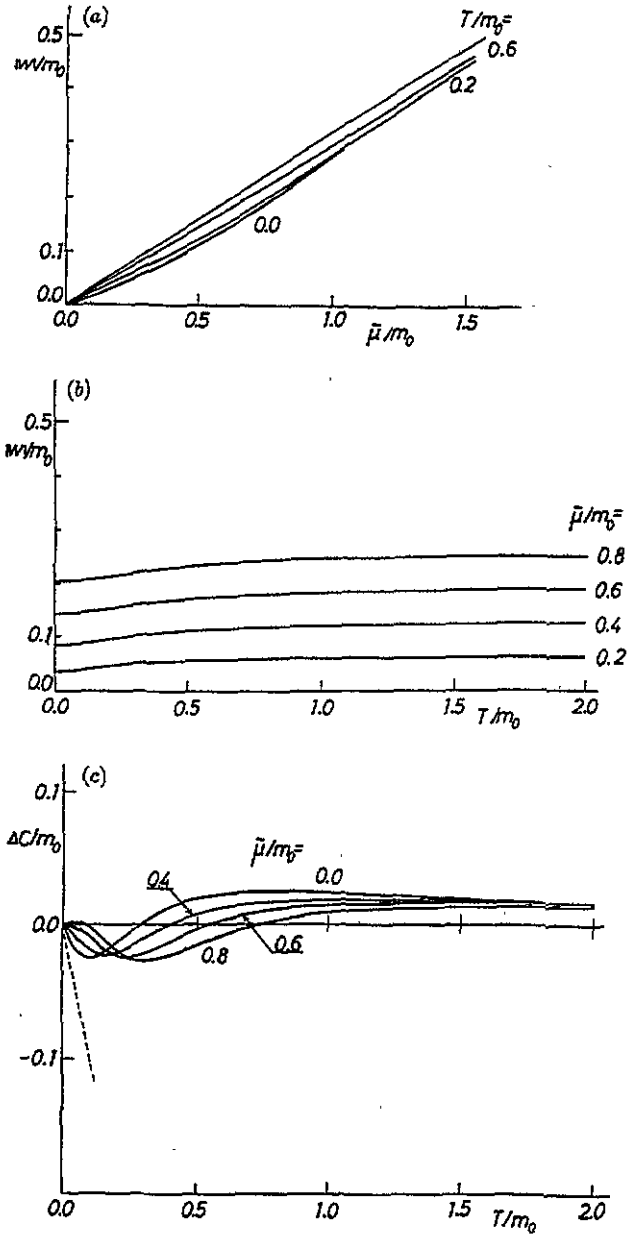


**Figure 3.** (*a*) Behaviour of the mean misfit as a function of the parameter $\tilde{\mu}$ at various temperatures $T/m_0 = 0$, 0.2 and 0.6 ($V_1 = -2/3$, $n_i/m_0 = 0.8$). The linear line $w = -\tilde{\mu}/\pi$ is that for the high-temperature limit. (*b*) Behaviour of the mean misfit as a function of temperature for a set of parameters $\tilde{\mu}/m_0 = 0.2$, 0.4, 0.6 and 0.8 ($V_1 = -2/3$, $n_i/m_0 = 0.8$). (*c*) Behaviour of the specific heat as a function of temperature for a set of parameters $\tilde{\mu}/m_0 = 0$, 0.4, 0.6 and 0.8 ($V_1 = -2/3$, $n_i/m_0 = 0.8$). Here the specific heat for a massless boson gas with no impurities is subtracted, $\Delta C = C - \pi T/3$.

to be 1/2, but enlarged scale (the right part) clearly shows that the exponent is not 1/2 but 3/2.

*2.1.2. $V_I < 0$.* For the attractive impurity potential $V_I < 0$, there is a gap energy $\omega_g$ for the impurity concentration $n_i$, which is smaller than the critical concentration $n_{ic}$ ($= (-A_I - 1)m_0$). Near the gap energy the expression of the mean misfit is the same as (30). The exponent of the deviation of $\tilde{\mu}$ from $\omega_g$ is 3/2. For low concentration of impurities, $U_g \sim U_{B1}$ and $a \sim (1 - U_{B1}^2)^{-3/4}(n_i/m_0)^{-1/2}$, where $U_{B1}$ is the normalized bound-state position, $U_{B1} = |1 - V_I^2/4|/(1 + V_I^2/4)$. Then the mean misfit becomes

$$\frac{w}{m_0} \sim -\frac{2}{3\pi}\left(\frac{n_i}{m_0}\right)^{1/4}(1 - U_{B1}^2)^{-9/8}\left(\frac{\tilde{\mu} - \omega_g}{m_0}\right)^{3/2}. \tag{32}$$

We show the behaviour of the mean misfit in figure 2(*a*). It shows that the exponent of the deviation of $\tilde{\mu}$ from $\omega_g$ is 3/2.

For the impurity concentration $n_i > n_{ic}$, the density of states at $\omega = 0$ is finite. The mean misfit shows a linear dependence on $\tilde{\mu}$ for smaller values of $\tilde{\mu}$:

$$w \sim -D_0\tilde{\mu}/\pi \tag{33}$$

where $D_0$ is the density of states at $\omega = 0$:

$$D_0 = [(1 + A_I + n_i/m_0)(1 - A_I - n_i/m_0)]^{1/2}. \tag{34}$$

We show the behaviour of the mean misfit in figure 3(*a*). It clearly shows the linear dependence on $\tilde{\mu}$ for smaller values of $\tilde{\mu}$.

### 2.2. *Finite-temperature properties of the mean misfit*

For the impurity potential $H_{imp}^I$, $w_1$ becomes

$$w_1 = -\frac{1}{\pi}\int D(\omega)[f(\omega - \tilde{\mu}) - f(\omega)]\,d\omega. \tag{35}$$

Using the symmetric property of the density of states $D(\omega) = D(-\omega)$, we obtain

$$w_1 = -\frac{1}{\pi}\int_0^\infty D(\omega)\frac{\sinh(\tilde{\mu}/T)}{\cosh(\tilde{\mu}/T) + \cosh(\omega/T)}\,d\omega. \tag{36}$$

The other parts $w_2$ and $w_3$ become zero. We show the behaviour of the mean misfit $w_1$ in figure 1(*a*) ($V_I > 0$) and figures 2(*a*) and 3(*a*) ($V_I < 0$) as a function of the natural misfit (the chemical potential $\tilde{\mu}$) at two temperatures $T/m_0 = 0.2$ and 0.6. For the case of finite gap energy, the mean misfit increases exponentially at lower temperature $T/m_0 = 0.2$ but linearly at higher temperature $T/m_0 = 0.6$ and approaches the linear line $w_1 = -\tilde{\mu}/\pi$ (figures 1(*a*) and 2(*a*)). For the concentration $n_i > n_{ic}$ (figure 3(*a*); $V_I = -2/3$, $n_i/m_0 = 0.8$), the mean misfit shows a linear dependence on $\tilde{\mu}$ for smaller values of $\tilde{\mu}$ and approaches the linear line $w_1 = -\tilde{\mu}/\pi$ with increase of temperature. In figures 1(*b*), 2(*b*) and 3(*b*), we show the mean misfit as a function of temperature for a set of parameters $\tilde{\mu}/m_0 = 0.2$, 0.4, 0.6, 0.8, 1.0, 1.2, 1.4 (figure 1(*b*)), $\tilde{\mu}/m_0 = 0.2$, 0.4, 0.6, 0.8, 1.0 (figure 2(*b*)) and $\tilde{\mu}/m_0 = 0.2$, 0.4, 0.6, 0.8 (figure 3(*b*)). For the case of finite gap energy (figures 1(*b*) and 2(*b*)), the curves tend to zero at zero temperature for smaller values of $\tilde{\mu}(< \omega_g)$ but tend to finite value for larger values of $\tilde{\mu}(> \omega_g)$. For the case of zero gap energy (figure 3(*b*)), the curves tend to finite values at zero temperature.

Next we will give the analytical expressions of the mean misfit for several limiting cases.

*2.2.1. Lower temperatures.* At lower temperatures we will use the density of states (29) estimated near the gap energy for $n_i < n_{ic}$,

$$D(\omega) \sim \frac{1}{a^{1/2}(1-U_g^2)^{3/2}} \left(\frac{|\omega|-\omega_g}{m_0}\right)^{1/2}.$$

The case of $V_i > 0$ corresponds to the infinite critical concentration $n_{ic} \to \infty$, and we always use this form for any concentration of impurities. The mean misfit is given by the following expressions for the three cases:

(i) $\tilde{\mu} < \omega_g$

$$w \sim -\frac{\sinh(\tilde{\mu}/T)T^{3/2}e^{-\omega_g/T}}{(\pi a m_0)^{1/2}(1-U_g^2)^{3/2}} \tag{37}$$

(ii) $\tilde{\mu} = \omega_g$

$$w \sim -\frac{\sinh(\tilde{\mu}/T)T^{3/2}e^{-\tilde{\mu}/T}}{(\pi a m_0)^{1/2}(1-U_g^2)^{3/2}} \frac{2-\sqrt{2}}{2}\zeta(3/2) \tag{38}$$

(iii) $\tilde{\mu} > \omega_g$

$$w \sim -\frac{1}{\pi(a m_0)^{1/2}(1-U_g^2)^{3/2}} \left(\frac{2}{3}(\tilde{\mu}-\omega_g)^{3/2} + \frac{\zeta(2)T^2}{2(\tilde{\mu}-\omega_g)^{1/2}} + \cdots\right). \tag{39}$$

Here $\zeta(s)$ is a zeta function and $\zeta(3/2) \simeq 2.612$, $\zeta(2) = \pi^2/6 \simeq 1.645$.

For $n_i > n_{ic}$, the density of states is estimated as $D(\omega) \sim D_0 + D_2\omega^2$. Then the mean misfit becomes

$$w \sim -D_0\tilde{\mu}/\pi - D_2\tilde{\mu}(\pi^2T^2 + \tilde{\mu}^2)/3\pi \tag{40}$$

which shows a linear dependence on $\tilde{\mu}$ for smaller values of $\tilde{\mu}$.

For the critical impurity concentration $n_i = n_{ic}$, the density of states is estimated as

$$D(\omega) \sim (\sqrt{3}/2)[-2(A_1+1)]^{1/3}(|\omega|/m_0)^{1/3}. \tag{41}$$

Then the mean misfit becomes

$$w \sim -\frac{\sqrt{3}}{2\pi}[-2(A_1+1)]^{1/3}\left(\frac{\tilde{\mu}}{m_0}\right)^{1/3}\tilde{\mu}\left[\frac{3}{4} + \frac{\pi^2}{18}\left(\frac{T}{\tilde{\mu}}\right)^2\right] \tag{42}$$

for $\tilde{\mu} \gg T$ and

$$w \sim -(\sqrt{3}/2\pi)[-2(A_1+1)]^{1/3}(T/m_0)^{1/3}\tilde{\mu}I \tag{43a}$$

$$I = \int_0^\infty \frac{x^{1/3}\,dx}{2\cosh^2(x/2)} \simeq 1.021 \tag{43b}$$

for $\tilde{\mu} \ll T$.

*2.2.2. Higher temperatures.* At higher temperatures it is convenient to use the following expression for the fermion number:

$$N(\tilde{\mu}, n_i, T) = LT \sum_n \frac{-iU_n}{(1+U_n^2)^{1/2}} + \frac{L}{\pi}(\Lambda + \tilde{\mu}) \tag{44}$$

where $U_n$ is determined in (13). At higher temperatures $T \gg m_0$, $n_i$, $|U_n|$ becomes large and is approximated as

$$U_n \sim (\omega_n - i\tilde{\mu} \pm n_i/A_1^2)/(m_0 + n_i/A_1) \tag{45}$$

where we take the upper sign + for $\omega_n > 0$ and the lower sign − for $\omega_n < 0$. Then the fermion number becomes

$$N(\tilde{\mu}, n_i, T) \sim \frac{LT}{2}\left(\frac{m_0 + n_i/A_1}{2\pi T}\right)^2 \times i\left[\psi^{(1)}\left(\frac{1}{2} + \frac{n_i/A_1^2 - i\tilde{\mu}}{2\pi T}\right) - \psi^{(1)}\left(\frac{1}{2} + \frac{n_i/A_1^2 + i\tilde{\mu}}{2\pi T}\right)\right] + \frac{L}{\pi}(\Lambda + \tilde{\mu}) \tag{46}$$

where $\psi^{(n)}(z)$ are poly-gamma functions. We will give the expressions for the mean misfit $w$ at higher temperatures in the two limiting cases:

(i) $\tilde{\mu} \ll T$

$$w \sim -\frac{\tilde{\mu}}{\pi}\left[1 + \frac{1}{2}\left(\frac{m_0 + n_i/A_1}{2\pi T}\right)^2 \psi^{(2)}\left(\frac{1}{2} + \frac{n_i/A_1^2}{2\pi T}\right)\right] \tag{47}$$

(ii) $\tilde{\mu} \gg T$

$$w \sim -\frac{\tilde{\mu}}{\pi}\left(1 - \frac{(m_0 + n_i/A_1)^2}{2\tilde{\mu}^2}\right). \tag{48}$$

### *2.3. Specific heat*

In this section we will discuss the specific heat of the system. In figures 1(*c*), 2(*c*) and 3(*c*) we show the behaviour of the specific heat as a function of temperature for a set of parameters (or chemical potentials) $\tilde{\mu}$. We have subtracted the specific heat for a massless boson gas with no impurity, $C = \pi T/3$, which is just the high-temperature limit of the system. Figures 1(*c*) ($V_1 = 1$, $n_i/m_0 = 0.1$, $\omega_g/m_0 \simeq 1.0040$) and 2(*c*) ($V_1 = -2/3$, $n_i/m_0 = 0.1$, $\omega_g/m_0 \simeq 0.4695$) are like each other and correspond to the case of finite gap energy. Figure 3(*c*) ($V_1 = -2/3$, $n_i/m_0 = 0.8$) corresponds to the case of zero gap energy.

Next we will give the analytical expressions for several limiting cases.

*2.3.1. Lower temperatures.* At lower temperatures we will use the density of states (29) estimated near the gap energy for $n_i < n_{ic}$. Then the specific heat per unit length is given by the following expressions for each case:

(i) $\tilde{\mu} < \omega_g$

$$C \sim \frac{1}{a^{1/2}(1-U_g^2)^{3/2}} \frac{(\omega_g - \tilde{\mu})^2}{2(\pi m_0 T)^{1/2}} e^{-(\omega_g - \tilde{\mu})/T} \tag{49}$$

(ii) $\tilde{\mu} = \omega_g$

$$C \sim \frac{1}{a^{1/2}(1-U_g^2)^{3/2}} \frac{15T^{3/2}}{8(\pi m_0)^{1/2}} \left(1 - \frac{1}{2\sqrt{2}}\right) \zeta(5/2) \tag{50}$$

(iii) $\tilde{\mu} > \omega_g$

$$C \sim \frac{1}{a^{1/2}(1-U_g^2)^{3/2}} \frac{2}{\pi} \left(\frac{\tilde{\mu} - \omega_g}{m_0}\right)^{1/2} T\zeta(2). \tag{51}$$

Here $\zeta(5/2) \simeq 1.341$.

For $n_i > n_{ic}$, the specific heat is estimated as

$$C \sim \pi D_0 T/3 + (\pi D_2 T/3)(\tilde{\mu}^2 + 7\pi^2 T^2/5). \tag{52}$$

For the critical impurity concentration $n_i = n_{ic}$, the specific heat is estimated as

$$C \sim (\sqrt{3}/2)[-2(A_1+1)]^{1/3}(\tilde{\mu}/m_0)^{1/3}\pi T/3 \tag{53}$$

for $\tilde{\mu} \gg T$ and

$$C \sim (\sqrt{3}/2)[-2(A_1+1)]^{1/3}(2T/\pi)(1-2^{-4/3})\Gamma(10/3)\zeta(7/3) \tag{54}$$

for $\tilde{\mu} \ll T$. Here $\Gamma(10/3) \simeq 2.778$ and $\zeta(7/3) \simeq 1.415$. In obtaining (54) we have used (41) for the density of states and the table in [18].

*2.3.2. Higher temperatures.* At higher temperatures it is convenient to use the method of Luttinger and Ward [19] to calculate the thermodynamic potential of the system. We expand the thermodynamic potential over $(m_0 + n_i/A_1)/T$ and differentiate this potential with respect to $T$. Then the entropy of the system per unit length becomes

$$S \sim \frac{\pi T}{3} + (m_0 + n_i/A_1)^2 \left\{ \frac{-1}{2\pi T} + \mathrm{Re}\left[ \frac{n_i/A_1^2 - i\tilde{\mu}}{4\pi^2 T^2} \psi^{(1)}\left(\frac{1}{2} + \frac{n_i/A_1^2 - i\tilde{\mu}}{2\pi T}\right)\right]\right\}. \tag{55}$$

From this entropy we get the specific heat of the system,

$$C \sim \frac{\pi T}{3} + \left(m_0 + \frac{n_i}{A_1}\right)^2 \left[ \frac{1}{2\pi T} - \frac{n_i/A_1^2}{2\pi^2 T^2} \psi^{(1)}\left(\frac{1}{2} + \frac{n_i/A_1^2}{2\pi T}\right) - \frac{(n_i/A_1)^2}{8\pi^3 T^3} \psi^{(2)}\left(\frac{1}{2} + \frac{n_i/A_1^2}{2\pi T}\right)\right] \tag{56}$$

for $\tilde{\mu} \ll T$ and

$$C \sim (\pi T/3)[1 + (m_0 + n_i/A_1)^2/(2\tilde{\mu}^2)] \tag{57}$$

for $\tilde{\mu} \gg T$.

## 3. Impurity potential $H^2_{\rm imp}$

### *3.1. Ground-state properties of the mean misfit*

At $T = 0$, the fermion number (equation (18)) is analytically calculated to become

$$N(\tilde{\mu}, n_{\rm i}, T=0) = \frac{L}{\pi}\left\{\mathrm{Im}\left[\tilde{\mu}\frac{U}{(1-U^2)^{1/2}} - \frac{m_0}{(1-U^2)^{1/2}} - \frac{n_{\rm i}}{2}\log\left(A_2 + \frac{U}{(1-U^2)^{1/2}}\right)^2\right] + n_{\rm i}\tan^{-1}\left(\frac{1}{A_2}\right) + \Lambda\right\}. \tag{58}$$

Here $U$ is determined by putting $\omega = 0$ in (16),

$$\tilde{\mu} = m_0 U + n_{\rm i}/[A_2 + U/(1-U^2)^{1/2}]. \tag{59}$$

Then the first part of the mean misfit at $T = 0$ is written as

$$\frac{w_1}{m_0} = -\frac{1}{\pi}\mathrm{Im}\left[\frac{\tilde{\mu}}{m_0}\frac{U(\tilde{\mu})}{[1-U^2(\tilde{\mu})]^{1/2}} - \frac{1}{[1-U^2(\tilde{\mu})]^{1/2}} - \frac{n_{\rm i}}{2m_0}\log\left(A_2 + \frac{U(\tilde{\mu})}{[1-U^2(\tilde{\mu})]^{1/2}}\right)^2 + \frac{1}{[1-U^2(0)]^{1/2}} + \frac{n_{\rm i}}{2m_0}\left(A_2 + \frac{U(0)}{[1-U^2(0)]^{1/2}}\right)^2\right]. \tag{60}$$

This behaviour is shown in figure 4(*a*) ($V_2 = -2/3, n_{\rm i}/m_0 = 0.15$). In figure 4(*a*), the impurity concentration $n_{\rm i}$ is assumed to be smaller than the critical concentration $n_{\rm ic}$, in which the density of states at $\omega = 0$ becomes zero (equations (51)–(53) and figure 5 in [20]). For $n_{\rm i} < n_{\rm ic}$, there are two gap energies $\omega_{\rm g1}(>0)$ and $\omega_{\rm g2}(<0)$. For $\omega_{\rm g2} < \omega < \omega_{\rm g1}$, the density of states is zero. Near $\tilde{\mu} \sim \omega_{\rm g1}$, the mean misfit is given by

$$\frac{w_1}{m_0} \sim -\frac{2}{3\pi}\frac{[(\tilde{\mu}-\omega_{\rm g1})/m_0]^{3/2}}{a^{1/2}(1-U^2_{\rm g1})^{3/2}} \tag{61}$$

where $a = -\frac{1}{2}\,\mathrm{d}^2(\omega/m_0)/\mathrm{d}U^2_{\rm g1}$ and $U_{\rm g1}$ is the corresponding value of the gap energy $\omega_{\rm g1}$. The exponent of the deviation of $\tilde{\mu}$ from $\omega_{\rm g1}$ is 3/2. For $n_{\rm i} > n_{\rm ic}$, the mean misfit shows a linear dependence on $\tilde{\mu}$ for smaller values of $\tilde{\mu}$,

$$w_1 \sim -D_0\tilde{\mu}/\pi \tag{62}$$

where $D_0$ is the density of states at $\omega = 0$. The second part of the mean misfit $w_2$ vanishes. The third part $w_3$ becomes

$$w_3 = -(n_{\rm i}/\pi)\tan^{-1}(1/A_2) \tag{63}$$

for $n_{\rm i} < n_{\rm ic}$ and

$$w_3 = -\frac{n_{\rm i}}{\pi}\tan^{-1}\left(\frac{1}{A_2}\right) + \frac{1}{\pi}\mathrm{Im}\left[\frac{m_0}{[1-U^2(0)]^{1/2}} + \frac{n_{\rm i}}{2}\log\left(A_2 + \frac{U(0)}{[1-U^2(0)]^{1/2}}\right)^2\right] \tag{64}$$

for $n_{\rm i} > n_{\rm ic}$. We note that $w_3$ is not zero, in contrast to the case of the $H^1_{\rm imp}$ potential. This shows that a mean misfit is induced by impurities. The behaviour of $w_3$ is shown in figure 4(*b*). For $n_{\rm i} > n_{\rm ic}$, $w_3$ deviates from the linear line. This deviation is proportional to $(n_{\rm i} - n_{\rm ic})^{3/2}$ at $n_{\rm i} \sim n_{\rm ic}$.

*3.2. Finite-temperature properties of the mean misfit*

For the impurity potential $H^2_{\rm imp}$, the first part of the mean misfit $w_1$ becomes

$$w_1 = -\frac{1}{\pi}\int D(\omega)[f(\omega-\tilde{\mu})-f(\omega)]\,{\rm d}\omega. \tag{65}$$

We show this behaviour in figure 4(*a*) ($V_2 = -2/3, n_{\rm i}/m_0 = 0.15$) as a function of $\tilde{\mu}$ at three temperatures $T/m_0 = 0.2$, 0.6 and 1.6. In this case of finite gap energies $\omega_{\rm g1}$ ($> 0$) and $\omega_{\rm g2}$ ($< 0$), the mean misfit increases exponentially at lower temperatures but linearly at higher temperatures and approaches the linear line $w = -\tilde{\mu}/\pi$. In figure 4(*c*) ($V_2 = -2/3, n_{\rm i}/m_0 = 0.15$) we show the mean misfit $w_1$ as a function of temperature for a set of parameters $\tilde{\mu}/m_0 = 0.2$, 0.4, 0.6, 0.8. For $\tilde{\mu} < \omega_{\rm g1}$ the mean misfit tends to zero as $T$ approaches zero. In figure 4(*d*) ($V_2 = -2/3$, $n_{\rm i}/m_0 = 0.15$), we show the second part of the mean misfit $w_2$ as a function of temperature. This part is related to the asymmetric property of the density of states, $D(-\omega) \neq D(\omega)$.

Next we will give the analytical expressions of the mean misfit for several limiting cases.

*3.2.1. Lower temperatures.* At lower temperatures we will use the density of states estimated near the gap energy $\omega_{\rm g1}$ for $n_{\rm i} < n_{\rm ic}$,

$$D(\omega) = \frac{1}{a^{1/2}(1-U^2_{\rm g1})^{3/2}}\left(\frac{\omega-\omega_{\rm g1}}{m_0}\right)^{1/2}. \tag{66}$$

Then the mean misfit is given by the following expressions for each case:

(i) $\tilde{\mu} < \omega_{\rm g1}$

$$w_1 \sim -\frac{T^{3/2}{\rm e}^{-\omega_{\rm g1}/T}}{(\pi a m_0)^{1/2}(1-U^2_{\rm g1})^{3/2}}\frac{{\rm e}^{\tilde{\mu}/T}-1}{2} \tag{67}$$

(ii) $\tilde{\mu} = \omega_{\rm g1}$

$$w_1 \sim -\frac{T^{3/2}}{(\pi a m_0)^{1/2}(1-U^2_{\rm g1})^{3/2}}\frac{1-{\rm e}^{-\tilde{\mu}/T}}{2}\frac{2-\sqrt{2}}{2}\zeta(3/2) \tag{68}$$

(iii) $\tilde{\mu} > \omega_{\rm g1}$

$$w_1 \sim -\frac{1}{\pi(a m_0)^{1/2}(1-U^2_{\rm g1})^{3/2}}\left(\tfrac{2}{3}(\tilde{\mu}-\omega_{\rm g1})^{3/2}+\frac{\zeta(2)T^2}{2(\tilde{\mu}-\omega_{\rm g1})^{1/2}}\right). \tag{69}$$

For the impurity concentration $n_{\rm i} > n_{\rm ic}$, the density of states is estimated as $D(\tilde{\mu})$. Then the mean misfit becomes

$$w_1 \sim -D(\tilde{\mu})\tilde{\mu}/\pi \tag{70}$$

which shows the linear dependence on $\tilde{\mu}$ for smaller values of $\tilde{\mu}$, since $D(\tilde{\mu}) \sim D_0$ for $\tilde{\mu} \sim 0$. For the critical impurity concentration $n_{\rm i} = n_{\rm ic}$, the density of states is given by (66) by putting $\omega_{\rm g1} = 0$. Then the mean misfit becomes

$$w_1 \sim -\frac{1}{\pi(a m_0)^{1/2}(1-U^2_{\rm g1})^{3/2}}\left(\tfrac{2}{3}\tilde{\mu}^{3/2}+\frac{\zeta(2)T^2}{2\tilde{\mu}^{1/2}}\right) \qquad (\tilde{\mu} \gg T) \tag{71a}$$

$$w_1 \sim -\frac{1}{\pi(a m_0)^{1/2}(1-U^2_{\rm g1})^{3/2}}\frac{\tilde{\mu}T^{1/2}}{2}J \qquad (\tilde{\mu} \ll T) \tag{71b}$$

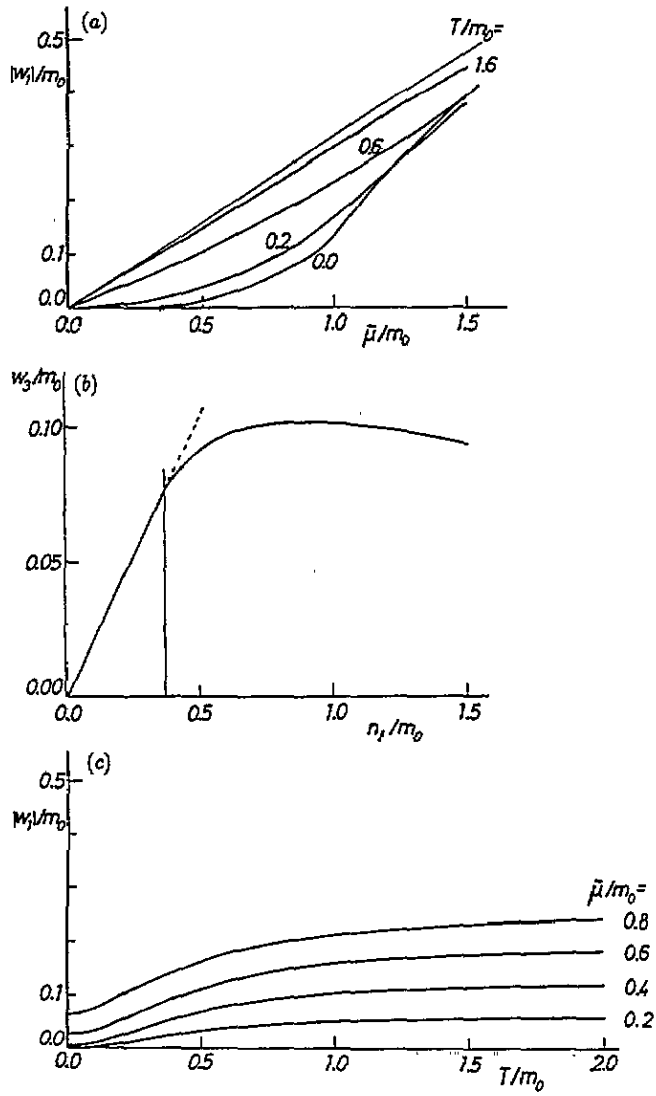


**Figure 4.** (*a*) Behaviour of the mean misfit $w_1$ as a function of the parameter $\tilde{\mu}$ at various temperatures $T/m_0 = 0$, 0.2, 0.6 and 1.6 ($V_2 = -2/3, n_i/m_0 = 0.15$). The linear line $w = -\tilde{\mu}/\pi$ is that for the high-temperature limit. (*b*) Behaviour of the mean misfit $w_3$ at $T = 0$ as a function of impurity concentration $n_i$ ($V_2 = -2/3$). (*c*) Behaviour of the mean misfit $w_1$ as a function of temperature for a set of parameters $\tilde{\mu}/m_0 = 0.2$, 0.4, 0.6 and 0.8 ($V_2 = -2/3, n_i/m_0 = 0.15$). (*d*) Behaviour of the mean misfit $w_2$ as a function of temperature ($V_2 = -2/3, n_i/m_0 = 0.15$). (*e*) Behaviour of the specific heat as a function of temperature for a set of parameters $\tilde{\mu}/m_0 = 0$, 0.2, 0.4, 0.6, 0.8, 1.0, 1.2 and 1.4 ($V_2 = -2/3, n_i/m_0 = 0.15$). Here the specific heat for a massless boson gas with no impurities is subtracted, $\Delta C = C - \pi T/3$.

where

$$J = \int_0^\infty \frac{dx\, x^{1/2}}{2\cosh^2(x/2)} \simeq 1.072.$$

It is possible to make analytical calculations for the second part of the mean misfit $w_2$ at low temperatures. For $n_i \leqslant n_{ic}$,

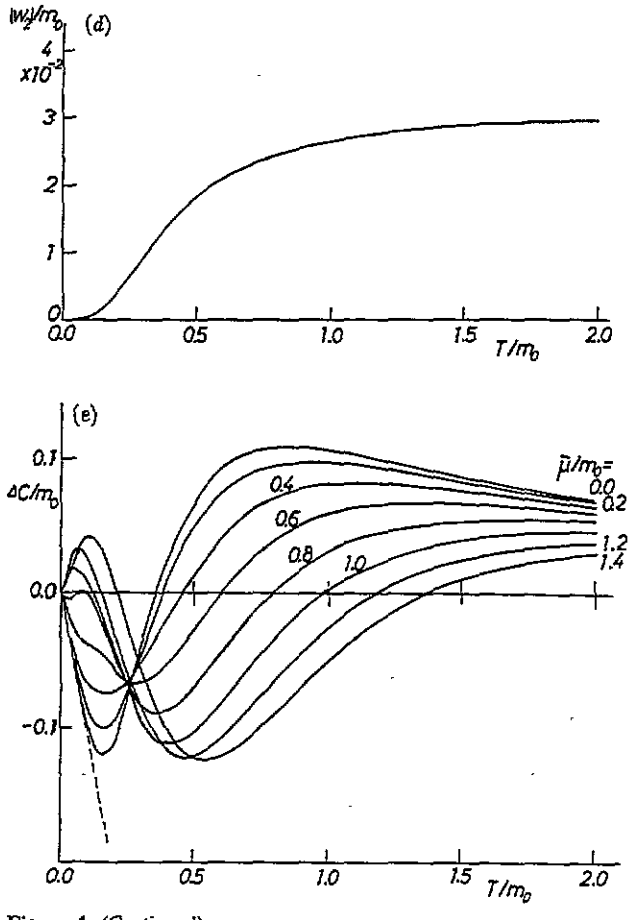


**Figure 4.** (Continued)

$$w_2 \sim -\frac{T^{3/2}e^{-\omega_{g1}/T}}{2(\pi m_0 a)^{1/2}(1-U_{g1}^2)^{3/2}} \tag{72}$$

and for $n_i > n_{ic}$,

$$w_2 \sim -\pi D_1 T^2/6 \tag{73}$$

where we have used the form $D(\omega) \sim D_0 + D_1\omega$ for the density of states at $\omega \sim 0$.

*3.2.2. Higher temperatures.* At higher temperatures it is convenient to use the following expression for the fermion number:

$$N(\tilde{\mu}, n_i, T) = LT\sum_n \frac{-iU_n}{(1+U_n^2)^{1/2}} + \frac{L}{\pi}(\Lambda + \tilde{\mu}) \tag{74}$$

where $U_n$ is determined in (14). At higher temperatures $T \gg m_0$, $|U_n|$ becomes large and is approximated as

$$U_n \sim [\omega_n - i\tilde{\mu} + in_i/(A_2 \pm i)]/m_0 \tag{75}$$

where we take the upper sign + for $\omega_n > 0$ and the lower sign − for $\omega_n < 0$. Then the fermion number becomes

$$N(\tilde{\mu}, n_i, T) \sim \frac{LT}{2}\left(\frac{m_0}{2\pi T}\right)^2 i\left\{\psi^{(1)}\left[\frac{1}{2}+\frac{1}{2\pi T}\left(\frac{in_i}{A_2+i}-i\tilde{\mu}\right)\right]\right.$$
$$\left.-\psi^{(1)}\left[\frac{1}{2}+\frac{1}{2\pi T}\left(\frac{-in_i}{A_2-i}+i\tilde{\mu}\right)\right]\right\}+\frac{L}{\pi}(\Lambda+\tilde{\mu}). \tag{76}$$

We will give the expressions of the mean misfit $w$ at high temperatures in the two limiting cases:

(i) $\tilde{\mu}, n_i \ll T$

$$w \sim -\frac{1}{\pi}\left[\frac{1}{2}\left(\frac{m_0}{2\pi T}\right)^2\left(\tilde{\mu}-\frac{n_i A_2}{A_2^2+1}\right)\psi^{(2)}(1/2)+\tilde{\mu}\right] \tag{77}$$

(ii) $\tilde{\mu} \gg T$

$$w \sim -\frac{1}{\pi}\left(\tilde{\mu}-\frac{1}{2}\frac{m_0^2}{\tilde{\mu}-n_i A_2/(A_2^2+1)}\right). \tag{78}$$

Here $\psi^{(2)}(1/2) = -14\zeta(3) \simeq -16.829$.

### *3.3. Specific heat*

In this section we will discuss the specific heat of the system. In figure 4(*e*) ($V_2 = -2/3$, $n_i/m_0 = 0.15$) we show the specific heat per unit length as a function of temperature for a set of misfit parameters $\tilde{\mu}/m_0 = 0$, 0.2, 0.4, 0.6, 0.8, 1.0, 1.2, 1.4. We have subtracted the specific heat for a massless boson gas of no impurities, $C = \pi T/3$, which is just the high-temperature limit of the system. In this case of finite gap energies ($\omega_{g1}/m_0 \simeq 0.343\,95$, $\omega_{g2} \simeq -1.008\,41$), figure 4(*e*) is similar to figure 1(*c*) or 2(*c*).

Next we will give the analytical expressions for several limiting cases.

*3.3.1. Lower temperatures.* At lower temperatures we use the density of states (66) for $n_i < n_{ic}$. The results are only to change $\omega_g \to \omega_{g1}$, $U_g \to U_{g1}$ in (49)–(51). For $n_i > n_{ic}$, the specific heat is estimated as

$$C \sim \pi D(\tilde{\mu})T/3 \tag{79}$$

with the density of states at $\omega = \tilde{\mu}$. For the critical concentration $n_i = n_{ic}$, the density of states is given by (66) by putting $\omega_{g1} = 0$, i.e.

$$D(\omega) = \frac{1}{a^{1/2}(1-U_{g1}^2)^{3/2}}\left(\frac{\omega}{m_0}\right)^{1/2}$$

for $\omega > 0$ and $D(\omega) = 0$ for $\omega < 0$. Then the specific heat per unit length is estimated as

$$C \sim \frac{1}{a^{1/2}(1-U_{g1}^2)^{3/2}}\left(\frac{\tilde{\mu}}{m_0}\right)^{1/2}\frac{\pi T}{3} \tag{80}$$

for $\tilde{\mu} \gg T$ and

$$C \sim \frac{1}{a^{1/2}(1-U_{g1}^2)^{3/2}}\frac{15T^{3/2}}{8(\pi m_0)^{1/2}}\left(1-\frac{1}{2\sqrt{2}}\right)\zeta(5/2) \tag{81}$$

for $\tilde{\mu} \sim 0$.

*3.3.2. Higher temperatures.* At higher temperatures we use the same method as in section 2.3.2. We expand the thermodynamic potential over $m_0/T$ and differentiate this potential with respect to $T$. Then the entropy per unit length becomes

$$S \sim \frac{\pi T}{3} + m_0^2 \left[\left[ \frac{-1}{2\pi T} + \mathrm{Re}\left\{ \frac{\mathrm{i}}{4\pi^2 T^2} \left( \frac{n_\mathrm{i}}{A_2 + \mathrm{i}} - \tilde{\mu} \right) \psi^{(1)} \left[ \frac{1}{2} + \frac{\mathrm{i}}{2\pi T} \left( \frac{n_\mathrm{i}}{A_2 + \mathrm{i}} - \tilde{\mu} \right) \right] \right\} \right] \right]. \tag{82}$$

The first term is the contribution from the free boson gas $m_0 = 0, n_\mathrm{i} = 0$. The specific heat per unit length is estimated as

$$C \sim \frac{\pi T}{3} + m_0^2 \left\{ \frac{1}{2\pi T} + \mathrm{Re}\left[ \frac{1}{2\pi^2 T^2} \frac{-\mathrm{i}n_\mathrm{i}}{A_2 + \mathrm{i}} \psi^{(1)} \left( \frac{1}{2} + \frac{1}{2\pi T} \frac{\mathrm{i}n_\mathrm{i}}{A_2 + \mathrm{i}} \right) \right.\right.$$
$$\left.\left. + \frac{1}{8\pi^3 T^3} \left( \frac{n_\mathrm{i}}{A_2 + \mathrm{i}} \right)^2 \psi^{(2)} \left( \frac{1}{2} + \frac{1}{2\pi T} \frac{\mathrm{i}n_\mathrm{i}}{A_2 + \mathrm{i}} \right) \right] \right\} \tag{83}$$

for $\tilde{\mu} \ll T$ and

$$C \sim (\pi T/3)[1 + m_0^2/(2\tilde{\mu}^2)] \tag{84}$$

for $\tilde{\mu} \gg T$.

## 4. Summary

We have calculated the mean misfit and the specific heat of the one-dimensional quantum sine–Gordon system with impurities for the special case of the quantum parameter, $g^2 = 4\pi$. Two types of impurities are considered. We have used the Fermi–Bose relation and the self-consistent $t$-matrix approximation for impurities. The effect of impurities appears at lower temperatures, especially at the ground state $T = 0$. One is the change of the exponent of the mean misfit from 1/2 in the pure system to 3/2 in our system with impurities.

For the impurity potential $H^1_\mathrm{imp}$, the overall behaviour of the mean misfit for the case of repulsive potential $V_\mathrm{I} > 0$ is similar to that of the pure system except just near $\tilde{\mu} = \omega_\mathrm{g}$. For the case of attractive impurity potential $V_\mathrm{I} < 0$, the behaviour of the mean misfit at $T = 0$ is different from that of the pure system, since the gap energy is reduced by the impurities. Moreover, with increasing concentration of impurities, the gap energy $\omega_\mathrm{g}$ vanishes and the density of states at $\omega = 0$ becomes finite. In this case the mean misfit starts from the origin and linearly increases near $\tilde{\mu} \sim 0$.

The above statements apply to the case of the impurity potential $H^2_\mathrm{imp}$ except that the mean misfit is induced by the impurities even when the misfit parameter $\tilde{\mu}$ is zero.

We note that the overall behaviours of the mean misfit $w_\mathrm{I}$ and the specific heat per unit length $C$ at finite temperatures depend on the existence of the gap. We have made analytical calculations of the mean misfit and the specific heat in the high- and low-temperature limits.

Here we comment on the case of a negative value of $\tilde{\mu}$, $\tilde{\mu} < 0$. For the $H^1_\mathrm{imp}$ potential we can give the same results except for sign as the positive case of $\tilde{\mu}$ from the symmetry of the density of states. For the $H^2_\mathrm{imp}$ potential the density of states is asymmetric, so our discussions are confined to the case of positive value of $\tilde{\mu}$ except for the analytical calculations in the high-temperature limit. However, it is easy to extend our discussions to the case of negative value of $\tilde{\mu}$.

The rewritten fermion systems (4), (7), (8) are related to the problems of one-dimensional Anderson localization [21, 22] except for the existence of the mass term. We have obtained the expression of the density of states by use of the $t$-matrix approximation for impurities

and calculated the specific heat and the mean misfit. It is possible that our self-consistent $t$-matrix approximation is inaccurate in one-dimensional problems. The density of states near the gap energy will not be expressed as $D(\omega) \sim (\omega - \omega_g)^{1/2}$. The critical exponent of the mean misfit at $T = 0$ may not be 3/2.

Here we try a simple argument on this problem. We will apply the discussion on the Schrödinger equation with $\delta$-function potentials by other authors [22] to our Dirac-type equations. For the repulsive potential $V_1 (> 0)$, there will be gap energy like the results of our $t$-matrix approximation. However, the critical exponent of the mean misfit will not be 3/2. For the attractive potential $V_1 (< 0)$ or the second type potential $V_2$, except for the special case over the distance between impurities [23], the gap energy will not exist. There will exist the edge of the band, which decreases exponentially. The critical concentration will become zero, $n_{ic} = 0$. There will be no phase transition as a function of the chemical potential or the natural misfit at $T = 0$. In this case our self-consistent $t$-matrix approximation will be valid only at intermediate temperatures—not too low temperatures. Also at low temperatures the results for the case $n_i > n_{ic}$ will be useful, since the critical concentration $n_{ic} = 0$.

Exact numerical calculations or Monte Carlo simulations for the density of states of our fermion systems are now under investigation.

## References


[1] Yamamoto H 1983 *Prog. Theor. Phys.* **70** 372
[2] Suzumura Y and Fukuyama H 1980 *J. Phys. Soc. Japan* **49** 915, 2081
[3] McMillan W L 1975 *Phys. Rev.* B **12** 1187
[4] Yoon J G, Kwag Y J, Cho Y S and Kwun S-I 1993 *J. Phys. Soc. Japan* **62** 327
[5] Nakanishi K 1979 *J. Phys. Soc. Japan* **46** 1434
[6] Karder M and Nelson D R 1985 *Phys. Rev. Lett.* **55** 1157
Karder M 1987 *Nucl. Phys.* B **290** 582
Tsvelik A M 1992 *Phys. Rev. Lett.* **68** 3889
[7] Villain J 1982 *J. Physique Lett.* **43** L551
[8] Rubinstein M, Shraiman B and Nelson D R 1983 *Phys. Rev.* B **27** 1800
[9] Nelson D R 1983 *Phys. Rev.* B **27** 2902
[10] Toner J 1991 *Phys. Rev. Lett.* **67** 2537
[11] Ioffe L B and Vinokur V M 1987 *J. Phys. C: Solid State Phys.* **20** 6149
[12] Yamamoto H 1993 *Phys. Lett.* A **177** 76
[13] Luther A and Emery V J 1974 *Phys. Rev. Lett.* **33** 589
[14] Luther A and Peschel I 1974 *Phys. Rev.* B **9** 2911
[15] Bergnoff H and Thacker H B 1979 *Phys. Rev.* D **19** 3666
[16] Korepin V E 1979 *Theor. Math. Phys.* **41** 169; 1980 *Commun. Math. Phys.* **76** 165
[17] Pokrovsky V L and Talapov A L 1979 *Phys. Rev. Lett.* **42** 65; 1980 *Sov. Phys.–JETP* **51** 134
[18] Gradshteyn I S and Ryzhik I M 1980 *Table of Integrals, Series, and Products* (New York: Academic) p 352
[19] Luttinger J M and Ward J C 1960 *Phys. Rev.* **118** 1417
[20] Yamamoto H 1994 *J. Low Temp. Phys.* **94** 77
[21] MacKinnon A and Kramer B 1983 *Z. Phys.* B **53** 1
Pichard J L and Sarma G 1985 *J. Phys. C: Solid State Phys.* **18** 3457
Abrahams E, Anderson P W, Licciardello D C and Ramakrishnan T V 1979 *Phys. Rev. Lett.* **42** 673
Abrikosov A A and Ryzhkin I A 1978 *Adv. Phys.* **27** 147
[22] Lax M and Phillips J C 1958 *Phys. Rev.* **110** 41
Frisch H L and Lloyd S P 1960 *Phys. Rev.* **120** 1175
Borland R E and Bird N F 1964 *Proc. Phys. Soc.* **83** 23
[23] Borland R E 1961 *Proc. Phys. Soc.* **78** 926
Clerk G J and McKellar B H J 1993 *Phys. Rev.* B **47** 6942